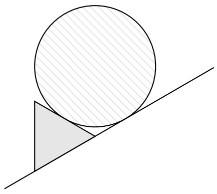
4001. Consider the curve $x = y^4 - 4y^2$.

(a) Show that the tangent to the curve is parallel to the y axis at three points, one of which is the origin.

(b) Sketch the curve, marking the points in (a) and all axis intercepts.

4002. A smooth cylindrical barrel of mass m and radius r is at rest on a slope of inclination 30°, held by a light chock in the form of an equilateral triangular prism of side length r. The contact between chock and slope is rough, with coefficient of friction μ.



Determine the least possible value of μ.

4003. Find all possible solution curves of

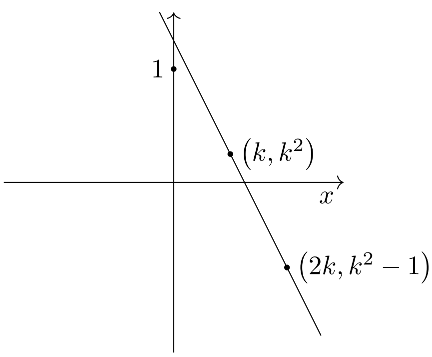
$$\left(\frac{dy}{dx} + 2x\right)\left(\frac{dy}{dx} - y\right) = 0.$$

4004. *Triangular numbers* have the form $T_n = \frac{1}{2}n(n+1)$, for $n \in \mathbb{N}$. Prove that $k \in \mathbb{N}$ is triangular if and only if $8k + 1$ is a square.

4005. Sketch $y = \cos^2 x$.

4006. Two variables y and z are related by the equation $y^2z - y + z = 1$. Show that y is never stationary with respect to z.

4007. Show that the y intercept of the line through (k, k^2) and $(2k, k^2 - 1)$, for $k \neq 0$, satisfies $y > 1$.



4008. Two consecutive maxima of the curve $y = \sin 4x$ are joined by a line segment. Show that the area enclosed by the curve and this line segment is $\frac{1}{2}\pi$.

4009. Show that $\displaystyle\lim_{x\to 0} \frac{1 - \cos 4x}{\sin 3x} = 0$.

4010. Show that the area of the region enclosed by the graphs $y = |x| + 2$ and $y = x^2$ is $6\frac{2}{3}$.

4011. A student is attempting to solve a trigonometric equation for $\theta \in [0, 2\pi)$. He writes

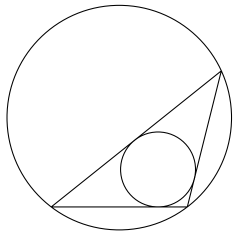
> Solving algebraically,
>
> $$\tan\left(2\theta + \tfrac{\pi}{2}\right) = \sqrt{3}$$
> $$\Longrightarrow 2\theta + \tfrac{\pi}{2} = ..., \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}, ...$$
> $$\Longrightarrow \theta = ..., \tfrac{-\pi}{6}, \tfrac{5\pi}{6}, \tfrac{11\pi}{6}, \tfrac{17\pi}{3}, ...$$
>
> So, in $[0, 2\pi)$, the solution is
>
> $$\theta = \tfrac{5\pi}{6}, \tfrac{11\pi}{6}, \tfrac{17\pi}{3}.$$

Explain the error, and correct it.

4012. Show that there is no real x which satisfies both of the inequalities $3 - |x - 1| \geq 0$ and $x^2 - 3x - 11 > 0$.

4013. Two graphs are given by equations $y = a + bx$ and $y = bx - ax^2$, where $a, b \in \mathbb{R}$ are constants. Show that, if the graphs intersect, then they have infinitely many points of intersection.

4014. A triangle has circles circumscribed and inscribed.



Prove that the ratio of radii is at least 2.

4015. A game is played with three dice. In the first turn, all three dice are rolled. Any die showing an odd number is then removed from the game. This same process is repeated until all of the dice have been removed. An example game is

Turn	1st	2nd	3rd
Dice rolled	3	1	1
Score(s)	$\{2, 5, 5\}$	$\{6\}$	$\{3\}$
Parity	EOO	E	O
Dice removed	2	0	1

In this question, the above is notated $(2, 0, 1)$.

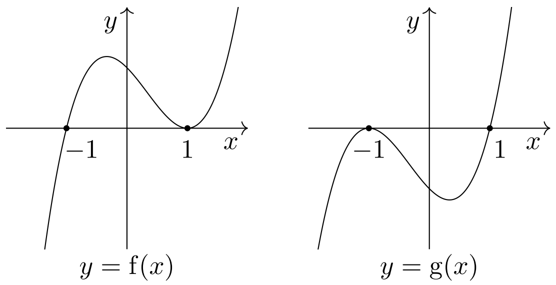
(a) Show that $\mathbb{P}(2, 0, 1) = \frac{3}{32}$.

(b) Show that there are 6 different games which finish after three turns.

(c) Hence, show that the probability p that the game ends after exactly three turns is $\frac{127}{512}$.

4016. Factorise $2x^4 - 11x^3 - 19x^2 - 11x - 21$ fully.

4017. Two monic cubic functions f and g have graphs as depicted. Each has either a single or double root at $x = \pm 1$.



Sketch the following graphs:

(a) $y = \mathrm{f}(x)\,\mathrm{g}(x)$,

(b) $y = \mathrm{f}(x) + \mathrm{g}(x)$,

(c) $y = \mathrm{f}(x) - \mathrm{g}(x)$.

4018. The *harmonic series* is defined as

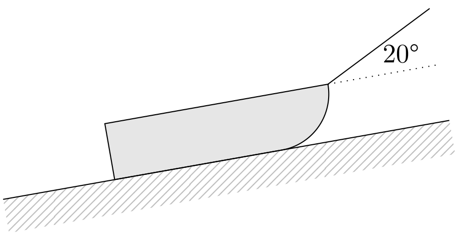
$$S_\infty = \sum_{r=1}^{\infty} \frac{1}{r}.$$

By grouping the series in successive sets of $1, 1, 2, 4, 8, \ldots$ terms, prove that S_∞ diverges.

4019. (a) Prove the identity $\tan^2\theta \equiv \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$.

(b) Hence, find $\displaystyle\int \frac{2\sec 2\theta}{\sec 2\theta + 1}\, d\theta$.

4020. A child is pulling a sledge of mass 5 kg up a hill, via a string. The slope is inclined at 10° to the horizontal, and the string is inclined at 20° to the slope.



While it is in motion, a frictional force of 25 N acts on the sledge. To begin with, the child pulls the sledge at a constant 1.2 ms^{-1}.

(a) Draw a force diagram for the sledge.

(b) Determine the tension in the string.

(c) The child now lets go of the string. The sledge moves a further distance d up the slope, before coming to rest. Determine d.

(d) What happens next?

4021. Two functions are related as follows:

$$\mathrm{f}(x) + \mathrm{g}(x) = \sqrt{2}\sin x$$
$$\mathrm{f}(x) - \mathrm{g}(x) = \sqrt{2}\cos x.$$

(a) Find $\mathrm{f}(x)$ and $\mathrm{g}(x)$ in terms of $\sin x$ and $\cos x$.

(b) Show that there is a constant k such that the following is an identity:

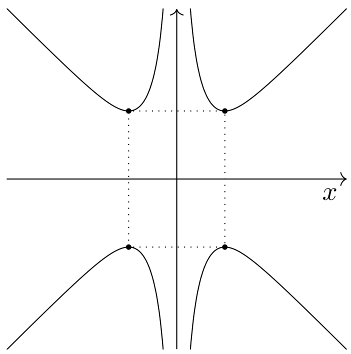
$$\mathrm{f}(x)^2 + \mathrm{g}(x)^2 \equiv k.$$

4022. It is suggested that, if $y = \mathrm{f}(x)$ is a solution curve of the differential equation

$$\frac{dy}{dx} + 4xy^2 = 0,$$

then solution curves of the form $y = A\,\mathrm{f}(x)$, for $A \in \mathbb{R}$, should also exist. Either prove or disprove this suggestion.

4023. The curve $x^2\left(y^2 - x^2\right) = 4$ has four SPs.



Show that these SPs are vertices of a rectangle whose sides are in the ratio $1 : \sqrt{2}$.

Bonus: This rectangle has the same dimensions as which everyday object?

4024. Two non-constant APs have nth terms a_n and b_n. Prove the following results:

(a) For any $p, q \in \mathbb{R}$, the sequence $pa_n + qb_n$ is arithmetic.

(b) For any $k \in \mathbb{R}$, the sequence k^{a_n} is geometric.

(c) The sequence a_n/b_n converges to the ratio of the common differences.

4025. A rigid unit square lamina is placed in the positive quadrant of a horizontal (x, y) plane, with three of its vertices on the axes. At the origin, a force of $\mathbf{i} + 2\mathbf{j}$ N acts on it, and at $(1, 1)$, a force of $-\mathbf{i}$ N. A third force acts, keeping the lamina in equilibrium. Determine the line of action and the magnitude of the third force.

4026. For constant $r \in \mathbb{R}$, show that

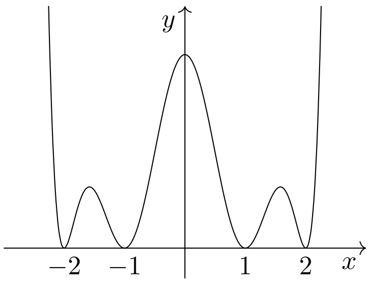
$$\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{r^2 - x^2}}\right) = \frac{r^2 + 2x^2}{\left(r^2 - x^2\right)^{\frac{5}{2}}}.$$

4027. A large set of continuous data has median 40.15, lower quartile 26.60 and upper quartile 45.31. A sample of ten data is taken from the set. Stating two assumptions you make, find the probability that

(a) no datum in the sample exceeds 45.31,

(b) exactly five data lie between the upper and lower quartiles.

4028. This graph is $y = (x^4 - 5x^2 + 4)^2$.

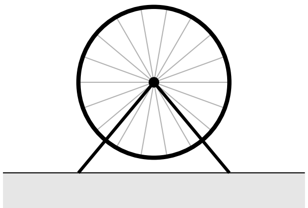


Sketch the graph $\sqrt{y} = x^4 - 5x^2 + 4$.

4029. Prove or disprove the following: "If a polynomial function f is such that $f(a) = f'(a) = 0$, then $f(x)$ has a factor of $(x - a)^2$."

4030. Show that the roots of $\tan^2\theta - \sec\theta = 1$, if they are represented on a unit circle, have rotational symmetry order 3.

4031. A ferris wheel, of mass 8 tonnes and diameter 6 metres, is supported on a pair of rigid struts, each of mass 200 kg, as shown.



The struts are 5 metres long, and the top of the ferris wheel is 7 metres off the ground. The ground exerts no moment on either strut, and the struts exert no horizontal force on the wheel.

(a) Draw force diagrams for the wheel and for the right-hand strut.

(b) Find, to the nearest kN, the magnitude of the force applied by the ground on each strut.

(c) The spokes of the ferris wheel are metal cables that are in tension. Assuming that the ferris wheel's mass is on its circumference, explain why the spokes at the top experience lower tension than the spokes at the bottom.

4032. A line segment is drawn through the points

$$A : (k+1, 2k),$$
$$B : (2k, k^2+1).$$

As $k \to 1$, both points tend to $(2, 2)$. Show that the gradient of AB tends to 0.

4033. Solve the equation $\left(\sqrt{x}\right)^5 + x - \dfrac{756}{\sqrt{x}} = 0$.

4034. The independent random variables X_1 and X_2 both have the distribution $X \sim B(4, 0.5)$. Find

(a) $\mathbb{P}(|X_1 - 2| = 1)$,

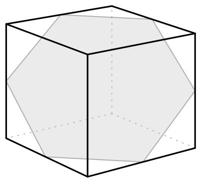
(b) $\mathbb{P}(X_1 = X_2)$.

4035. Using the reverse chain rule, find $\displaystyle\int \frac{1}{x \ln x}\, dx$.

4036. Solve, for $x, y \in [0, \pi)$,

$$\sqrt{3}\sin x - \sin y = 0,$$
$$\sqrt{3}\cos x - \cos y = 2.$$

4037. The diagram shows a cube of unit side length, and a regular hexagon joining the midpoints of six of the cube's edges.



Six of the cube's edges are chosen at random. Show that the probability that their midpoints lie at the vertices of such a regular hexagon is $\frac{1}{231}$.

4038. The implicit relation $e^x + e^y = 2$ defines a curve.

(a) Show that $y = -x$ is tangent to the curve.

(b) Show that $x = \ln 2$, $y = \ln 2$ are asymptotes.

(c) Hence, sketch the curve.

4039. Find a and b such that, for small θ in radians,

$$\frac{\cos\theta}{\cos\theta + 1} \approx a + b\theta^2.$$

4040. In this question, functions f and g are defined over $\mathbb{R}$, and 0 is not in the range of g.

One of the following statements is true; the other is not. Prove the one and disprove the other.

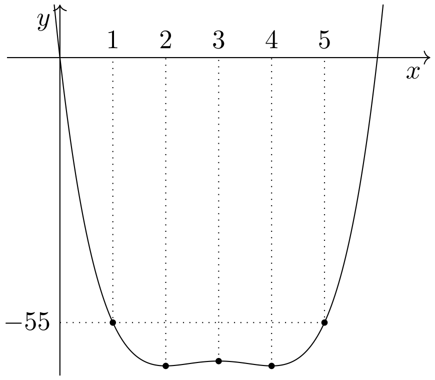
(a) $f(x) = 0 \implies \dfrac{f(x)}{g(x)} = 0$,

(b) $\lim\limits_{x\to 0} f(x) = 0 \implies \lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0$.

4041. Three dice have been rolled, giving scores X, Y, Z. Show that $\mathbb{P}(X+Y=Z \mid X+Y+Z=6)=\frac{1}{5}$.

4042. Find $\displaystyle\int \tan^2 x\,dx$.

4043. A quartic curve has the following properties: it passes through the origin, $(1,-55)$ and $(5,-55)$, and it has stationary points at $x=2$, $x=3$ and $x=4$.



Find the equation of the curve.

4044. Two particles have positions, in 1D, given by

$$x_1 = t^4 - t^3,$$
$$x_2 = 20t - 32.$$

(a) Show that the particles collide.

(b) Show that, when they do, their relative speed is instantaneously zero.

4045. Find the average value of $|\sin\theta|$, for $\theta \in \mathbb{R}$.

4046. A function g is defined, for constants p, q, r, by

$$\mathrm{g}(x) = x^3 - px^2 + qx - r.$$

The equation $\mathrm{g}(x)=0$ has three distinct real roots $x = 1, k, k^2$. Show that

$$p = \frac{1-k^3}{1-k}.$$

4047. A curve is defined, over the domain $(-\pi, \pi)$, by the equation $y = \operatorname{cosec}^2 x + \sin x + 3$.

(a) Find the coordinates of any axis intercepts and stationary points, and the equations of any asymptotes.

(b) Hence, sketch the curve.

4048. Factorise $3003x^3 + 1896x^2y + 51xy^2 - 90y^3$.

4049. Find the following indefinite integral, writing your answer as a simplified fraction:

$$\int \frac{2x^7}{(x^4-1)^{\frac{3}{2}}}\,dx.$$

4050. Prove that, if a straight line $y=\mathrm{f}(x)$ is tangent to a positive quartic graph $y=\mathrm{g}(x)$ at two distinct points, then $\mathrm{f}(x)-\mathrm{g}(x) \le 0$ for all $x \in \mathbb{R}$.

4051. Using the definition $\cot\theta = \frac{\cos\theta}{\sin\theta}$, and assuming the compound-angle formulae for sin and cos (but not tan), prove that

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}.$$

4052. Sketch the curve $\sqrt{x} + y^2 = 1$.

4053. A computer generates X values randomly from the set $\{X \in \mathbb{R} : 0 \le X \le 10\}$, equally distributed across the interval. The variable Y is then defined by $Y = 10X - X^2$. Show that $\mathbb{P}(Y > 16) = 0.6$.

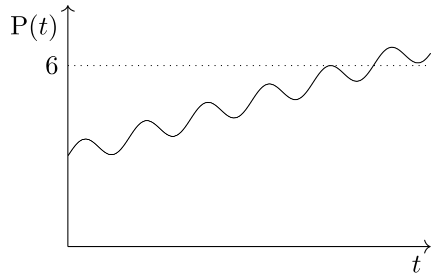
4054. If $x_1, x_2 = \dfrac{\sqrt[3]{a} \pm 1}{\sqrt[3]{a} \mp 1}$, show that $\bar{x} = \dfrac{\sqrt[3]{a^2}+1}{\sqrt[3]{a^2}-1}$.

4055. Prove that the product of five consecutive integers is divisible by 120.

4056. A population of rodents, measured in thousands, is modelled as having size

$$\mathrm{P}(t) = 3 + 0.3t + 0.4\sin 3.1t,$$

where the sine function takes inputs in radians, and t is measured in years.



(a) Write down the initial population.

(b) By considering the range of the sine function, show that the population does not reach 6000 before $t = \frac{26}{3}$.

(c) Show that, at the first maximum after this time, the population falls short of 6000, but that, at the following maximum, it exceeds 6000.

(d) Find, to 2dp, the time at which the population first reaches 6000.

4057. The product $P = \displaystyle\prod_{i=2}^{\infty}\left(1 - \frac{1}{i^2}\right)$ is given by

$$P = \lim_{N\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{N^2}\right).$$

Prove that $P = \frac{1}{2}$.

4058. Solve the following pair of simultaneous equations:

$$(x+y-1)(x+y+1)=0$$
$$(x-y-1)(x-y+1)=0$$

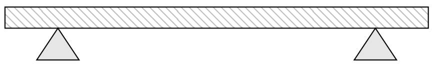
4059. A family of circles is defined, for $\theta \in \mathbb{R}$, by

$$(x-2\cos\theta)^2+(y-2\sin\theta)^2=1.$$

On a sketch, shade the points of the (x, y) plane which lie on at least one of the circles.

4060. Show that the graph $2x^2+5xy+2y^2=1$ has no stationary points.

4061. A uniform beam, length 8 metres and mass 20 kg, rests horizontally on two supports, each of which is positioned 1 metre from an end of the beam.



Show that there are regions of the beam on which a 70 kg person cannot stand without tipping it.

4062. A graph is given, over a suitable domain, as

$$y=\ln\big(\tan(x+\tfrac{\pi}{4})\big).$$

(a) Show that $\dfrac{dy}{dx}=\cot\big(x+\frac{\pi}{4}\big)+\tan\big(x+\frac{\pi}{4}\big)$.

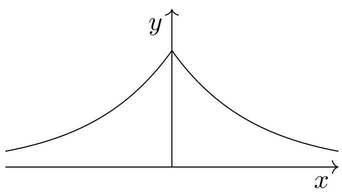
(b) Hence, show that $\dfrac{dy}{dx}=\dfrac{2\tan^2 x+2}{1-\tan^2 x}$.

(c) Hence, show that $\dfrac{dy}{dx}=2\sec 2x$.

4063. (a) Show that the parabola $y=x^2-x$ and the cubic $y=x^3-3x^2+2x$ are normal to each other at one of their points of intersection.

(b) Sketch the curves on the same set of axes.

4064. The graph below shows $y=\mathrm{f}(x)$. The function f has even symmetry, and is defined by $\mathrm{f}(x)=e^{1-2x}$ for positive x.



Show that $\displaystyle\int_{-\infty}^{\infty}\mathrm{f}(x)\,dx$ is finite, and evaluate it.

4065. A set S of 100 idealised data is given by

$$\underbrace{0,0,...,0}_{n},\underbrace{1,1,...,1}_{100-n}.$$

Two data are chosen at random from S, without replacement. Their numerical values are modelled with random variables S_1 and S_2. It is given that $\mathbb{P}(S_1S_2=0)=\frac{19}{55}$. Find n.

4066. A mathematical chemist is modelling an industrial process. Its temperature θ oscillates sinusoidally, except when a positive feedback process is induced. Then, the chemist proposes that the process obeys the differential equation

$$\frac{d^2\theta}{dt^2}-2\frac{d\theta}{dt}+5\theta=39e^{4t}.$$

The chemist produces a solution

$$\theta=3e^{4t}+5e^{t}\cos 2t.$$

(a) Verify that the solution satisfies the equation.

(b) Explain why the long-term behaviour of the system cannot be as the model suggests.

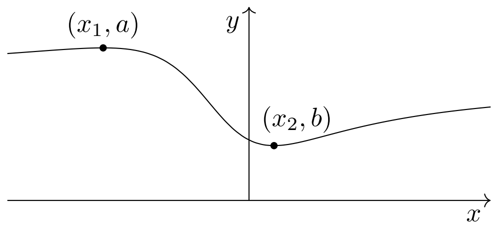
4067. Solve the following equation:

$$\frac{\sin x}{1-\cos x}+\frac{\cos x}{1-\sin x}=0.$$

4068. A function is defined over the reals by

$$\mathrm{f}(x)=\frac{x^2+1}{x^2+x+2}.$$

Part of the graph of $y=\mathrm{f}(x)$ is shown below, with stationary points whose y values are a and b.



Show that $a-b=\dfrac{4\sqrt{2}}{7}$.

4069. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning polynomial functions f and g:

① $\mathrm{f}'(x)\equiv\mathrm{g}(x)$,

② $\mathrm{f}(x)\equiv\displaystyle\int_0^x \mathrm{g}(t)\,dt$.

4070. A chord is drawn to the curve $y=x^3+kx$, where k is a constant, at points $x=\pm p$. Prove that chord and curve enclose two regions of equal area.

4071. In a card game, hands of four cards are dealt from a standard deck. Find the probability that any particular hand contains exactly three suits.

4072. The *kinetic energy* of a particle is defined, in terms of mass m kg and speed v ms^{-1}, by

$$T = \tfrac{1}{2}mv^2.$$

The *gravitational potential energy* is defined as

$$V = mgh,$$

where g ms^{-2} is gravitational acceleration and h m is height above an arbitrary baseline.

Prove that, when a projectile is thrown vertically upwards at initial speed u, total energy $T + V$ is conserved in the ensuing motion.

4073. By considering a derivative at $x = 1$, show that

$$\int_1^2 \cos(\ln x)\, dx < 1.$$

4074. Events X, Y, Z have

$$\mathbb{P}(X) = 0.4, \quad \mathbb{P}(Y) = 0.5, \quad \mathbb{P}(Z) = 0.6.$$

Determine all possible values of the following. Give your answers in interval set notation.

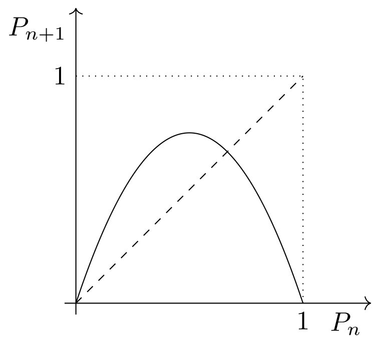
(a) $\mathbb{P}(X \cap Y \cap Z)$

(b) $\mathbb{P}(Y \cap Z \mid X)$

4075. A biologist models the size of a population of bees with a *logistic iteration*

$$P_{n+1} = \lambda P_n(1 - P_n),$$

where $\lambda > 0$ is a constant representing the rate of reproduction. The population is modelled as a fraction of some maximum, i.e. $P_n \in [0, 1]$ for all $n \in \mathbb{N}$.



(a) i. Determine the range of $x \mapsto \lambda x(1 - x)$ on the domain $[0, 1]$.

ii. Hence, show that $\lambda \le 4$.

(b) Let P be a fixed point of population.

i. Show that $P = 0$ or $P = 1 - \frac{1}{\lambda}$.

ii. Show that, if $\lambda < 1$, then the only stable state for the population is extinction.

iii. Show that population can never be stable with $P > \frac{3}{4}$.

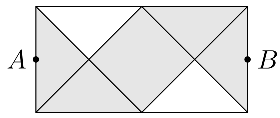
4076. Function f has instruction

$$\mathrm{f}(x) = (1 - 2x)^{-\frac{1}{2}}.$$

By expanding and substituting $x = \frac{1}{32}$, show that

$$\sqrt{15} \approx \tfrac{8192}{2115}.$$

4077. The seven regions of the schematic map below are shaded, or left blank, at random. In the example, five contiguous regions have been shaded.



Find the probability that points A and B end up connected by contiguous shaded regions. (Regions are contiguous if they share a border, but not if they only share a vertex.)

4078. A solid cylinder is described by the inequalities

$$(x-2)^2 + (y-6)^2 \le 4,$$
$$0 \le z \le 5.$$

(a) The parametric equation of the cylinder's axis of symmetry is expressed as

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

i. Write down the values of a_1, a_2, b_1, b_2.

ii. Explain why $b_3 \ne 0$.

iii. Explain why a_3 may be any number.

(b) Calculate the volume of the cylinder.

4079. Prove that, if $\tan \frac{1}{2}x$ is positive, then

$$\tan \tfrac{1}{2}x \equiv \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

4080. A function f is defined over $\mathbb{R}$ by

$$\mathrm{f} : x \mapsto x^4 + 4x^3 + 5x^2 + 2x + 2.$$

Show that there exists a constant $k \in \mathbb{R}$ for which

$$\mathrm{f}(k - x) \equiv \mathrm{f}(k + x).$$

4081. Points A, B, C are defined as

$$A : (q, q^2 + q),$$
$$B : (q^2, q),$$
$$C : (29, 0).$$

It is given that, for some $k \in \mathbb{R}$, $\overrightarrow{AC} = k\overrightarrow{BC}$. Find all possible values of q..

4082. In this question, do not use a calculator.

Solve the simultaneous equations:

$$\begin{aligned} x+y+z&=4,\\ 3x+2y+z&=13,\\ x-y-z&=0. \end{aligned}$$

4083. Find the equation of the circle which is tangent to the following curve at four distinct points:

$$(2x-1)^2\,(2y-1)^2=1.$$

4084. Two transformations T_1 and T_2 of the (x,y) plane are defined as follows. They are stretches, both fixing the origin, with scale factors and directions

	s.f.	Direction
T_1	3	parallel to the x axis
T_2	4	parallel to the y axis.

The generic vector $\mathbf{a}=p\mathbf{i}+q\mathbf{j}$ is transformed by a composition of T_1 and T_2.

(a) Find the image $\mathbf{b}$ of $\mathbf{a}$.

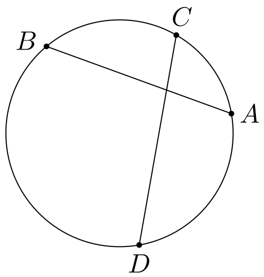
(b) Show that no vector other than the zero vector is left unchanged by the composition.

(c) Prove that the composition of T_1 and T_2 can not be expressed as either

i. a single two-dimensional enlargement, or

ii. a single one-dimensional stretch.

4085. Four points A,B,C,D are chosen at random on the circumference of a circle.



Find the probability that, as in the outcome above, the chord AB intersects the chord CD.

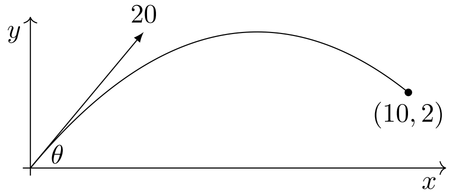
4086. Prove that, for any starting value $u_0\notin\{0,1\}$, the following sequence is periodic with period 3:

$$u_{n+1}=1-\frac{1}{u_n}.$$

4087. The curve $y=x^4-x^2$ is transformed by a stretch in the x direction and a stretch in the y direction. Its image is $y=ax^4-4ay^2$, where a is a positive constant. Find, in terms of a, the area scale factor of this transformation.

4088. Take $g=10$ in this question.

A particle is projected from the origin of a vertical (x,y) plane at speed 20 ms^{-1}. It is aimed at an angle θ above the horizontal so as to hit a target at the point $(10,2)$.



(a) Show that $5\tan^2\theta-40\tan\theta+13=0$.

(b) Find the possible angles of projection.

4089. An *Euler brick* is a cuboid with integer edge lengths, whose face diagonals are also of integer length. Show that, if (p,q,r) is a Pythagorean triple, then the cuboid with the following edge lengths is an Euler brick:

$$\begin{aligned} a&=p(4q^2-r^2),\\ b&=q(4p^2-r^2),\\ c&=4pqr. \end{aligned}$$

4090. By differentiating, verify that, for $\tan\frac{x}{2}\geq 0$,

$$\int \operatorname{cosec} x\,dx=\ln\left(\tan\tfrac{x}{2}\right)+c.$$

4091. Prove that three distinct primes cannot form a GP.

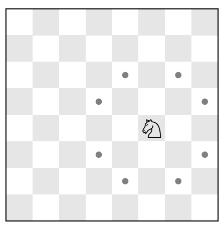
4092. In a simulation, the variables $\{X_1,X_2,X_3,X_4\}$ are binary, taking values in $\{0,1\}$. To begin with, they are all set to zero.

A random number from the set $\{0,1,2,3,4\}$ is then chosen, which gives the number of X_i variables available for change. Each available variable has a probability of $\frac{1}{2}$ of changing to 1.

Find the probability that exactly two X_i variables change to 1.

4093. Sketch the curve $y=x^2+\dfrac{1}{x^2}$.

4094. In chess, knights threaten squares as shown.



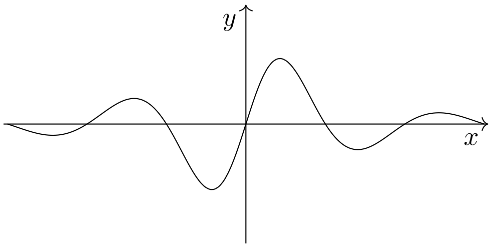
Two knights are placed at random on two different squares. The probability that they threaten each other is p. Show that $p={}^{1}\!/_{12}$.

4095. Coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ have these properties:

- $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$,
- $\mathbf{a}$ and $\mathbf{b}$ are perpendicular,
- $\mathbf{a}$ and $\mathbf{c}$ are parallel,
- $|\mathbf{b}| = 1$.
- $\mathbf{d} = -2\mathbf{i}$.

Find all possible vectors $\mathbf{b}$.

4096. The graph below shows $y = \dfrac{\sin x}{1 + x^2}$.



Show that the curve has infinitely many SPs.

4097. A quartic function f is invertible over each of the domains $(-\infty, \alpha]$ and $[\alpha, \infty)$. Also, $\mathrm{f}(\alpha) = 0$. Prove that $\mathrm{f}(x)$ has a factor of $(x - \alpha)^2$.

4098. Show that there are no values $x \in \mathbb{R}$ such that

$$e^{3x} = e^x - e^{-x}.$$

4099. A function f satisfies

$$x\,\mathrm{f}'(x) = (x + 1)\big(\mathrm{f}(x)\big)^2.$$

Show that, for any $a, b \in \mathbb{R}$,

$$\mathrm{f}(a)\big(b - a - \ln|a|\big) = 1.$$

4100. Show that there are 453600 possible arrangements of the letters of the word LOVELINESS.

——— END OF 41ST HUNDRED ———